

THE NUMBER OF PSEUDO-ANOSOV ELEMENTS IN THE MAPPING CLASS GROUP OF A FOUR-HOLED SPHERE

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ABSTRACT. We compute the growth series and the growth functions of reducible and pseudo-Anosov elements of the pure mapping class group of the sphere with four holes with respect to a certain generating set. We prove that the ratio of the number of pseudo-Anosov elements to that of all elements in a ball with center at the identity tends to one as the radius of the ball tends to infinity.

1. INTRODUCTION

A finitely generated group can be seen as a metric space after fixing a finite generating set. The metric is the so called word metric. As is well-known, the mapping class group of a compact surface is finitely generated, thus a metric space.

The purpose of this note is to prove that, after fixing a certain set of generators, in a ball centered at the identity in the pure mapping class group of a four holed sphere (which is a free group of rank two), almost all elements are pseudo-Anosov. More precisely, in a ball with center at the identity, the ratio of the number of pseudo-Anosov elements to the number of all elements tends to one as the radius of the ball tends to infinity. In fact, we prove more: We give the growth series of reducible and of pseudo-Anosov elements with respect to a fixed set of generators. It turns out that the growth functions of these elements are rational. This gives a partial answer to Question 3.13 and verifies Conjecture 3.15 in [2] in a special case.

2. PRELIMINARIES

Let G be a finitely generated group with a finite generating set A , so that every element of G can be written as a product of elements in $A \cup A^{-1}$. The *length* of an element $g \in G$ (with respect to A) is defined as

$$\|g\|_A = \min\{k : g = a_1 a_2 \cdots a_k, a_i \in A \cup A^{-1}\}.$$

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The *distance* between two elements g and h is defined as $d_A(g, h) = \|h^{-1}g\|_A$. The function d_A is a metric on G , called the *word metric*. Of course, this metric depends heavily on the generating set. The choice of different generating sets give rise to equivalent metrics. We will always fix a finite generating set A and drop A from the notation.

For a subset P of G , the *growth series* of P relative to the generating set A is the formal power series $\sum c_n x^n$, where the coefficient c_n of x^n is the number of elements of length n in P . The *growth function* of P is the function represented by the growth series. In the mapping class group, we may take P to be periodic, reducible or pseudo-Anosov elements.

Let S be a compact connected orientable surface of genus g with $r \geq 0$ holes (= boundary components). The *mapping class group* $\text{Mod}(S) = \text{Mod}(g, r)$ of S is defined as the group of isotopy classes of orientation-preserving homeomorphisms $S \rightarrow S$. The subgroup $\text{PMod}(g, r)$ of $\text{Mod}(g, r)$ consisting of isotopy classes of homeomorphisms preserving each boundary component of S is the *pure mapping class group*.

Thurston's classification of surface diffeomorphisms says that for a mapping class f which is not the identity exactly one of the followings holds: (1) f is periodic, i.e. $f^m = 1$ for some $m \geq 2$; (2) f is reducible, i.e. there is a (closed) one-dimensional submanifold C of S such that $f(C) = C$; (3) f is pseudo-Anosov (Anosov if S is a torus).

It is well known that the mapping class group $\text{Mod}(1, 0)$ of a torus is isomorphic to $SL(2, \mathbb{Z})$. The elements of the group $\text{Mod}(1, 0)$ are classified by the traces of the corresponding matrices; if f is an element of $\text{Mod}(1, 0)$, then it is periodic if $|\text{trace}(f)| < 2$, reducible if $|\text{trace}(f)| = 2$, and Anosov if $|\text{trace}(f)| > 2$ (cf. see [1]). In [5], Takasawa computed the growth series of periodic, reducible and Anosov elements of $\text{Mod}(1, 0)$ and found their growth functions. He proved that almost all elements of the mapping class group of the torus are Anosov. That is, with respect to a certain generating set, the ratio of the number of Anosov elements to the number of all elements in a ball centered at the identity tends to one as the radius of the ball tends to infinity.

Now let S be a sphere with four holes and let a and b be two distinct nonisotopic simple closed curves on S such that each of a and b separates S into two pairs of pants and that a intersects b precisely at two points (c.f. Figure 1). It is well known that $\text{PMod}(0, 4)$ is isomorphic to the free group F_2 and freely generated by the Dehn twists t_a and t_b about a and b respectively. We will always take this generating set below.

3. THE NUMBER OF REDUCIBLE AND PSEUDO-ANOSOV ELEMENTS IN THE MAPPING CLASS GROUP $\text{PMod}(0, 4)$

3.1. Counting certain elements in the free group of rank two. We begin by counting certain type of elements in the free group of rank two.

Let F_2 be the free group of rank two freely generated by $\{\alpha, \beta\}$. We fix this set of generators throughout this subsection.

The next lemma is elementary and is easy to prove.

Lemma 3.1. *The growth series of F_2 is*

$$h(x) = 1 + 4x + 4 \cdot 3x^2 + 4 \cdot 3^2x^3 + \cdots + 4 \cdot 3^{n-1}x^n + \cdots.$$

For an element $\gamma \in F_2$, let $C(\gamma, n)$ denote the set of elements in F_2 of length n of the form $w\gamma^kw^{-1}$, where k is an integer and $w \in F_2$. Let $|C(\gamma, n)|$ denote the cardinality of $C(\gamma, n)$.

Lemma 3.2. (1) *If $w\alpha^kw^{-1}$ and $v\alpha^lv^{-1}$ are reduced, then $w\alpha^kw^{-1} = v\alpha^lv^{-1}$ if and only if $w = v$ and $k = l$.*

(2) *For each nonnegative integer r , $|C(\alpha, 2r+1)| = |C(\alpha, 2r+2)| = |C(\beta, 2r+1)| = |C(\beta, 2r+2)| = 2 \cdot 3^r$.*

(3) *For each nonnegative integer r , $|C(\alpha\beta, 2r+1)| = 0$ and $|C(\alpha\beta, 2r+2)| = 4 \cdot 3^r$.*

Proof. If $w\alpha^kw^{-1} = v\alpha^lv^{-1}$ then $\alpha^{k-l} = w^{-1}v\alpha^lv^{-1}w\alpha^{-l}$, a commutator. Hence, $k = l$. Now, by looking at the lengths of each side of $\alpha^k = w^{-1}v\alpha^kv^{-1}w$, we deduce that $w = v$. The converse is clear, proving (1).

Define a function $\phi: C(\alpha, 2r+1) \rightarrow C(\alpha, 2r+2)$ by

$$\phi(w\alpha^kw^{-1}) = \begin{cases} w\alpha^{k+1}w^{-1}, & \text{if } k > 0 \\ w\alpha^{k-1}w^{-1}, & \text{if } k < 0, \end{cases}$$

where $w\alpha^kw^{-1}$ is reduced. Clearly, the function ϕ is onto. It follows from (1) that it is also one-to-one. Consider also the automorphism $\psi: F_2 \rightarrow F_2$ given by $\psi(\alpha) = \beta$ and $\psi(\beta) = \alpha$. The map ψ is an isometry and $\psi(C(\alpha, n)) = C(\beta, n)$. Thus, the first three equalities in (2) are proved. In order to complete the proof of (2), we show $|C(\alpha, 2r+1)| = 2 \cdot 3^r$. The proof of this claim is by induction on r .

Note that if k is even then the length of $w\alpha^kw^{-1}$ is even for any $w \in F_2$. Hence, $C(\alpha, 2r+1)$ contains the conjugates of odd powers of α . Note also that if $w\alpha^kw^{-1}$ is a reduced word of length n , then $-n \leq k \leq n$.

The set $C(\alpha, 1)$ contains only two elements, α and α^{-1} . Hence, the claim holds in the case $r = 0$.

Assume that $|C(\alpha, 2r+1)| = 2 \cdot 3^r$. Define a function φ from $C(\alpha, 2r+1)$ to the subsets of $C(\alpha, 2r+3)$ as follows:

- $\varphi(\alpha^{2r+1}) = \{\alpha^{2r+3}, \beta\alpha^{2r+1}\beta^{-1}, \beta^{-1}\alpha^{2r+1}\beta\};$
- $\varphi(\alpha^{-(2r+1)}) = \{\alpha^{-(2r+3)}, \beta\alpha^{-(2r+1)}\beta^{-1}, \beta^{-1}\alpha^{-(2r+1)}\beta\};$
- $\varphi(\alpha w \alpha^{-1}) = \{\alpha^2 w \alpha^{-2}, \beta \alpha w \alpha^{-1} \beta^{-1}, \beta^{-1} \alpha w \alpha^{-1} \beta\};$
- $\varphi(\alpha^{-1} w \alpha) = \{\alpha^{-2} w \alpha^2, \beta \alpha^{-1} w \alpha \beta^{-1}, \beta^{-1} \alpha^{-1} w \alpha \beta\};$
- $\varphi(\beta w \beta^{-1}) = \{\beta^2 w \beta^{-2}, \alpha \beta w \beta^{-1} \alpha^{-1}, \alpha^{-1} \beta w \beta^{-1} \alpha\};$
- $\varphi(\beta^{-1} w \beta) = \{\beta^{-2} w \beta^2, \alpha \beta^{-1} w \beta \alpha^{-1}, \alpha^{-1} \beta^{-1} w \beta \alpha\}.$

It is easy to check that the set

$$\{\varphi(x) : x \in C(\alpha, 2r+1)\}$$

is a partition of $C(\alpha, 2r + 3)$. That is, elements of this set are pairwise disjoint and their union is equal to $C(\alpha, 2r + 3)$. We deduce from this that $|C(\alpha, 2r + 3)| = 3|C(\alpha, 2r + 1)| = 2 \cdot 3^{r+1}$, completing the proof of (2).

It is clear that $|C(\alpha\beta, 2r + 1)| = 0$ for all $r \geq 0$. Note that for any $w \in F_2$, the word length of $w(\alpha\beta)^k w^{-1}$ is at least $2|k|$. That is, the set $C(\alpha\beta, 2r + 2)$ does not contain any conjugate of $(\alpha\beta)^k$ for $|k| > r + 1$.

The element $(\beta\alpha)^n$ is conjugate to $(\alpha\beta)^n$ and any element in $C(\alpha\beta, 2r + 2)$ is of the form $w(\alpha\beta)^n w^{-1}$ or $w(\beta\alpha)^n w^{-1}$ for some $w \in F_2$ with $\|w\| = r + 1 - n$. Hence, we will only consider the (reduced) words in these two forms.

The only conjugates of $(\alpha\beta)^k$ for $|k| = r + 1$ contained in $C(\alpha\beta, 2r + 2)$ are elements of

$$A_{r+1} = \{(\alpha\beta)^{r+1}, (\beta\alpha)^{r+1}, (\alpha\beta)^{-(r+1)}, (\beta\alpha)^{-(r+1)}\}.$$

All other elements of $C(\alpha\beta, 2r + 2)$ are conjugates of $(\alpha\beta)^k$ for $|k| \leq r$, hence they are conjugates of elements of $C(\alpha\beta, 2r)$.

Consider the subset of $C(\alpha\beta, 2r)$ consisting of the conjugates of $(\alpha\beta)^{\pm r}$. They form the set

$$A_r = \{(\alpha\beta)^r, (\beta\alpha)^r, (\alpha\beta)^{-r}, (\beta\alpha)^{-r}\}.$$

Each element of A_r gives rise two elements of length $2r + 2$ by conjugation. For instance, one may conjugate $(\alpha\beta)^r$ only with α and β^{-1} in order to get an element of length $2r + 2$. Therefore, there are eight such elements in $C(\alpha\beta, 2r + 2)$.

The elements of the difference $C(\alpha\beta, 2r) - A_r$ are of the form $\alpha w \alpha^{-1}$, $\alpha^{-1} w \alpha$, $\beta w \beta^{-1}$ or $\beta^{-1} w \beta$. The number of such elements is $|C(\alpha\beta, 2r)| - 4$ and each gives rise to three elements of length $2r + 2$ by conjugation (if there is cancellation, we do not need to take them).

It follows that

$$|C(\alpha\beta, 2r + 2)| = 4 + 8 + 3(|C(\alpha\beta, 2r)| - 4) = 3|C(\alpha\beta, 2r)|.$$

Now, (3) follows from the fact that $C(\alpha\beta, 2)$ consists of four elements; namely

$$C(\alpha\beta, 2) = \{\alpha\beta, \beta\alpha, (\alpha\beta)^{-1}, (\beta\alpha)^{-1}\}.$$

This finishes the proof of the lemma. \square

Corollary 3.3. *The number of elements of length n conjugate to a power of α , β or $\alpha\beta$ is $4 \cdot 3^r$ if $n = 2r + 1$ and $8 \cdot 3^r$ if $n = 2r + 2$ ($r \geq 0$).*

Proof. The set of elements of length n conjugate to the given elements is $C(\alpha, 2r + 1) \cup C(\beta, 2r + 1)$ if $n = 2r + 1$ and $C(\alpha, 2r + 2) \cup C(\beta, 2r + 2) \cup C(\alpha\beta, 2r + 2)$ if $n = 2r + 2$. These sets are pairwise disjoint. The result now follows from Lemma 3.2. \square

3.2. The mapping class group $\text{PMod}(0, 4)$. Since $\text{PMod}(0, 4)$ is isomorphic to F_2 , there are no periodic elements in $\text{PMod}(0, 4)$. Elements of $\text{PMod}(0, 4)$ different from the identity are either reducible or pseudo-Anosov. In this section, we compute the growth series and the growth functions of these elements in $\text{PMod}(0, 4)$.

Let S be a sphere with four holes. A simple closed curve a on S is called *trivial* if either it bounds a disc or it is parallel to a boundary component. Otherwise, it is called *nontrivial*.

Let us fix two nontrivial simple closed curves a and b on S intersecting transversely twice as in Figure 1. It is well known that the Dehn twists t_a and t_b generate the group $\text{PMod}(0, 4)$ freely. By the lantern relation, there is a unique simple closed curve c on S separating S into two pairs of pants and intersecting both a and b twice such that the Dehn twists t_a, t_b and t_c satisfy $t_a t_b t_c = 1$ (c.f. Figure 1). Thus, we have $t_c = (t_a t_b)^{-1}$, and hence conjugates of powers t_a, t_b and $t_a t_b$ are reducible. In fact, they are the only reducible elements in $\text{PMod}(0, 4)$.

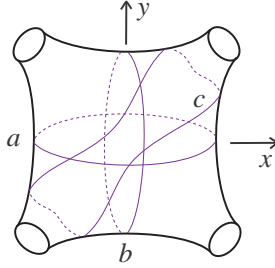


FIGURE 1. The Dehn twists about a, b, c satisfy $t_a t_b t_c = 1$ in $\text{PMod}(0, 4)$ by the lantern relation.

Lemma 3.4. *The reducible elements of $\text{PMod}(0, 4)$ consist of conjugates of nonzero powers of t_a, t_b and $t_a t_b$.*

Proof. Let f be a reducible element $\text{PMod}(0, 4)$. Then $F(d) = d$ for some nontrivial simple closed curve d and $F \in f$. Thus, $t_d f = f t_d$, since $f t_d f^{-1} = t_{F(d)} = t_d$. Since $\text{PMod}(0, 4)$ is a nonabelian free group and t_d can be completed to a free basis of $\text{PMod}(0, 4)$, we conclude that $f = t_d^k$ for some nonzero integer k .

It follows from the classification of simple closed curves on S (c.f. see [4]) that there is a homeomorphism $H : S \rightarrow S$ preserving each boundary component of S such that $H(d) \in \{a, b, c\}$.

Let h denote the isotopy class of H in $\text{PMod}(0, 4)$. If $H(d) = a$ then $f = t_a^k = h^{-1} t_a^k h$, if $H(d) = b$ then $f = h^{-1} t_b^k h$, and if $H(d) = c$ then $f = h^{-1} t_c^k h = h^{-1} (t_a t_b)^{-k} h$, proving the lemma. \square

We are now ready to state and prove the main result of this paper.

Theorem 3.5. *With respect to the generating set $\{t_a, t_b\}$ of $\text{PMod}(0, 4)$,*

(1) *the growth series of reducible elements is*

$$\begin{aligned} r(x) = & 4(x + 3x^3 + 3^2x^5 + 3^3x^7 + \cdots + 3^r x^{2r+1} + \cdots) \\ & + 8(x^2 + 3x^4 + 3^2x^6 + 3^3x^8 + \cdots + 3^r x^{2r+2} + \cdots). \end{aligned}$$

Hence, the growth function of reducible elements is

$$r(x) = \frac{4x + 8x^2}{1 - 3x^2}.$$

(2) *the growth series of pseudo-Anosov elements is*

$$4 \sum_{r=0}^{\infty} 3^r (3^{r+1} - 2) x^{2r+2} + 4 \sum_{r=1}^{\infty} 3^r (3^r - 1) x^{2r+1}$$

and the growth function of pseudo-Anosov elements is

$$p(x) = \frac{4x^2(1 + 3x)}{(1 - 3x)(1 - 3x^2)}.$$

(3) *if p_n and h_n denote the number of pseudo-Anosov and all elements of length at most n respectively, then we have*

$$\lim_{n \rightarrow \infty} \frac{p_n}{h_n} = 1.$$

Proof. By Lemma 3.4, reducible elements in $\text{PMod}(0, 4)$ are conjugates of nonzero powers of t_a, t_b and $t_a t_b$. By Corollary 3.3, the number of such elements of length $n > 0$ in $\text{PMod}(0, 4)$ is $4 \cdot 3^r$ if $n = 2r + 1$ and $8 \cdot 3^r$ if $n = 2r + 2$.

Therefore the growth series of reducible elements is

$$\begin{aligned} r(x) = & 4x + 4 \cdot 3x^3 + 4 \cdot 3^2x^5 + 4 \cdot 3^3x^7 + \cdots + 4 \cdot 3^r x^{2r+1} + \cdots \\ & + 8x^2 + 8 \cdot 3x^4 + 8 \cdot 3^2x^6 + 8 \cdot 3^3x^8 + \cdots + 8 \cdot 3^r x^{2r+2} + \cdots \\ = & (4x + 8x^2)(1 + 3x^2 + 3^2x^4 + 3^3x^6 + \cdots + 3^r x^{2r} + \cdots). \end{aligned}$$

It follows that the growth function is given by

$$r(x) = \frac{4x + 8x^2}{1 - 3x^2}.$$

This proves (1).

The growth series and the growth function of all elements are

$$\begin{aligned} h(x) = & 1 + 4x + 4 \cdot 3x^2 + 4 \cdot 3^2x^3 + \cdots + 4 \cdot 3^{n-1}x^n + \cdots \\ = & \frac{1 + x}{1 - 3x}. \end{aligned}$$

The growth series of pseudo-Anosov elements follows from this and (1). The growth function of pseudo-Anosov elements is

$$\begin{aligned} p(x) &= h(x) - 1 - r(x) \\ &= \frac{4x}{1-3x} - \frac{4x+8x^2}{1-3x^2} \\ &= \frac{4x^2(1+3x)}{(1-3x)(1-3x^2)}. \end{aligned}$$

This proves (2).

Let r_n denote number of reducible elements of length at most n . By (1), we have

$$\begin{aligned} r_n &= 4(1+3+3^2+\cdots+3^r) + 8(1+3+3^2+\cdots+3^{r-1}) \\ &= 10 \cdot 3^r - 6 \end{aligned}$$

if $n = 2r + 1$ and

$$\begin{aligned} r_n &= 4(1+3+3^2+\cdots+3^{r-1}) + 8(1+3+3^2+\cdots+3^{r-1}) \\ &= 2 \cdot 3^{r+1} - 6 \end{aligned}$$

if $n = 2r$. By Lemma 3.1, we get

$$\begin{aligned} h_n &= 1 + 4(1+3+3^2+\cdots+3^{n-1}) \\ &= 2 \cdot 3^n - 1. \end{aligned}$$

It follows that

$$\lim_{n \rightarrow \infty} \frac{r_n}{h_n} = 0.$$

Since $p_n = h_n - r_n - 1$, the proof of (3) follows. \square

3.3. A little more. Let ι (resp. j) denote the isotopy class of the rotation about the x -axis (resp. y -axis) by π . (We assume that the surface lie in the three space and is invariant under these rotations, as in Figure 1.) Let Γ denote the subgroup of the mapping class group $\text{Mod}(0, 4)$ generated by $\text{PMod}(0, 4)$, ι and j . Then Γ is isomorphic to $\text{PMod}(0, 4) \times \mathbb{Z}_2 \times \mathbb{Z}_2$, and is of index 6 in $\text{Mod}(0, 4)$.

Since ι, j and ιj preserve each nonboundary parallel simple closed curve up to isotopy, it can be shown that an element f in $\text{PMod}(0, 4)$ is pseudo-Anosov if and only if $f\iota, fj$ and $f\iota j$ are pseudo-Anosov. It follows that, with respect to the generating set $\{t_a, t_b, \iota, j\}$ of Γ , the ratio of the number of pseudo-Anosov elements to that of all elements in a ball of radius n centered at the identity tends to one as n tends to infinity. It would be good to extend this result to $\text{Mod}(0, 4)$ and to all $\text{Mod}(0, n)$.

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